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Title of Research Paper: Bounds For The Regularity Of Generalized Edge Ideals

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Abstract. The Castelnuovo–Mumford regularity is a fundamental invariant in commutative algebra. In this paper, we establish upper bounds for the regularity of powers of generalized edge ideals. Our results extend existing work and highlight new connections between combinatorial structures and algebraic invariants.

Keywords: Bounds, Regularity, Generalized Edge Ideals, combinatorial structures and algebraic invariants

1. Introduction

The notion of Castelnuovo–Mumford regularity plays a central role in modern commutative algebra and algebraic geometry, as it provides a quantitative measure of the complexity of graded modules and their associated sheaves. In particular, regularity governs the vanishing of cohomology, the degrees of generators of syzygies, and the computational behavior of minimal free resolutions. Over the years, considerable effort has been devoted to understanding and bounding the regularity of homogeneous ideals and graded modules in terms of intrinsic algebraic invariants.

A fundamental problem in this direction is to determine effective upper bounds for the regularity in terms of data such as the degrees of generators, the dimension, or other homological invariants. Classical results show that regularity is finite and behaves well under several natural operations, yet explicit bounds are often difficult to obtain and tend to grow rapidly with the complexity of the underlying structure. This has led to a rich body of work exploring both general bounds and sharper estimates in special cases, including monomial ideals, ideals with fixed dimension, and classes of ideals arising from combinatorial or geometric constructions.

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The following lemma gives us a nice description of the powers of the generalized edge ideals in the presence of a leaf vertex.

Lemma 3.1. Let G be a simple graph and $x_i \in V(G)$ a leaf vertex with $N_G(x_i) = \{x_j\}$. Suppose that $I = I_g(G)$ is a generalized edge ideal of G, $J = I_g(G \setminus x_i)$. Then for all $s \geq 2$ we have the following.

(i) $(I^s, x_i^{w_{i,j}} x_j^{w_{j,i}}) = (J^s, x_i^{w_{i,j}} x_j^{w_{j,i}})$;

(ii) If $w_{j,i} \leq w_{j,t}$ for all $x_t \in N_G(x_j)$, then $(I^s : x_i^{w_{i,j}} x_j^{w_{j,i}}) = I^{s-1}$.

The next result gives a bound for the regularity of powers of generalized edge ideals in terms of the regularity of two different ideals obtained from $I_g(G)^s$.

Theorem3.2 Let G be a simple graph and $x_i \in V(G)$ a leaf vertex with $N_G(x_i) = \{x_j\}$. Suppose that $I = I_g(G)$ is a generalized edge ideal of G, $J = I_g(G \setminus x_i)$, and $w_{j,i} \leq w_{j,t}$ for all $x_t \in N_G(x_j)$.

Then for all $s \geq 2$

$$reg(I^s) \leq w_{i,j} + w_{j,i} + \max\{reg(I^{s-1}), reg(J^s) - 1\}$$

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Finally, we give one of the main results of this article, which gives a bound for the regularity of powers of generalized edge ideal $I_g(G)$ in terms of the regularity of the ideal $I_g(G)$ and the regularity of various powers of the generalized edge ideals of a subgraph obtained from G.

Theorem3.4 Let G be a simple graph and $x_i \in V(G)$ a leaf vertex with $N_G(x_i) = \{x_j\}$. Suppose that $I = I_g(G)$ is a generalized edge ideal of G, $J = I_g(G \setminus x_i)$, and $w_{j,i} \leq w_{j,t}$ for all $x_t \in N_G(x_j)$.

Then for all $s \geq 2$

$$reg(I^s) \leq (s-1)(w_{i,j} + w_{j,i}) + \max\{reg(I), reg(J^2), reg(J^3), \ldots, reg(J^s).$$

In this paper, we investigate new bounds on the regularity of the generalized edge ideals. Our approach is based on combinatorial arguments, which allows us to derive bounds that improve upon or complement existing results in the literature. In particular, we establish 3.4 , providing new insight into how regularity depends on the structural properties of the object under consideration.

Our paper is organized as follows. In Section 2, we collect necessary definitions and terminologies. In Section 3, we have proved some results on the regularity. In Section 3, we give an upper bound for reg(I^s), where $I = I_g(G)$ is a generalized edge ideal, G contains a leaf vertex, and $s \geq 2$.

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2. Preliminaries

In this section, we recall some preliminary definitions, notations, and terminology of algebra and graph theory that will be used throughout the paper. We refer the reader to the standard texts [6], [46] for the definitions and results used in this paper. We recall some well-known notions of graph theory, and the associated algebraic objects that we want to study in the following sections.

Definition 2.1. Let G be a simple graph. An edge ideal $I(G)$ is defined by

$$I(G) := (x_i x_j : \{x_i, x_j\} \in E(G)).$$

Definition 2.2. Let G be a simple graph. A generalized edge ideal $I_g(G)$ of G is defined by

$$I_G(G) := (x_i^{w_{i,j}} x_j^{w_{j,i}} : \{x_i x_j\} \in E(G))),$$

where $w_{i,j} \geq 1$ is called the relative weight of x_i with respect to x_i.

Let $I_g(G)$ be a generalized edge ideal of a simple graph G and $x_i \in V(G)$. We say that the relative weight of x_i is uniform if all the relative weights of x_i are equal, i.e, there exist $w_i \in \mathbb{N}$ such that $w_{i,j} = w_i$ for all j with $\{x_i, x_j\} \in E(G)$.

Definition 2.3. Let G be a simple graph. A vertex $x \in V(G)$ is called a leaf vertex if $|N_G(x)| = 1$, where $N_G(x) = \{y \in V(G) : \{x, y\} \in E(G)\}$.

Let $R = \mathbb{K}[x_x, \ldots, x_n]$, and M be a graded R-module. We write G(M) to denote a set of minimal generators of the module M, and $\mu(M)$ to denote the cardinality of the set $G(M)$. A graded minimal free resolution of M is given by

$$\mathcal{F}: \cdots \to \bigoplus R(-j)^{\beta_{i,j}(M)} \xrightarrow{d(i,M)} \bigoplus R(-j)^{\beta_{i-1,j}(M)} \cdots \to \bigoplus R(-j)^{\beta_{0,j}(M)} \xrightarrow{d(0,M)} M \to 0$$

The numbers $\beta_{i,j}(M)$ are called the *graded Betti numbers* of the module M, and $\beta_j(M) = \sum_{j>0} \beta_{i,j}(M)$, $i \geq 0$ are called the *total Betti numbers* of the module M. Note that the maps

$d(i,M)$ in the minimal free resolution of M are between free R-modules, and hence they can be expressed as matrices with entries coming from R. Moreover, if I is a monomial ideal, the entries of the matrices $d(i,I)$ can be taken to be monomials as well. For a monomial, $f = x_1^{a_1} \ldots x_n^{a_n}$ the set Supp(f) = $\{x_i : a_i \neq 0\}$ is called the *support* of f. If Supp(f) = $\{x_{i_1}, \ldots, x_{i_k}\}$, we sometimes say that the monomial f is supported by the variables $x_{i_1}, \ldots, x_{i_k}$.

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Definition 2.4. For a graded R-module M, the Castelnuovo-Mumford regularity(for simply, the regularity) of M, denoted by reg$M$, is defined by

$$regM = \max\{j - i : \beta_{i,j}(M) \neq 0\}$$

In general, it is a very difficult task to get hold of the graded Betti numbers for a module M. But if M is given to be a monomial ideal, then one can compute the multigraded Betti numbers of M by looking at the upper-Koszul simplicial complexes associated with M. A simplicial complex Δ on the vertex set $V = \{x_i,\dots,x_n\}$ is a collection of subsets of V such that $\{x_i\} \in \Delta$ for all $x_i \in V$ and if $F \in \Delta$, $F' \subseteq F$ then $F' \in \Delta$. Elements of Δ are called the *faces* of Δ, and the maximal elements with respect to inclusion are called the *facets*.

Definition 2.5. Let $I \subseteq R = \mathbb{K}[x_1, \dots, x_n]$ be a monomial ideal and let $\alpha = (\alpha_1,\dots,\alpha_n) \in \mathbb{N}^n$ be an $\mathbb{N}^n$-graded degree. The *upper-Koszul simplicial complex* associated with I of degree α, denoted by $K^\alpha(I)$, is the simplicial complex over $V = \{x_1, \dots, x_n\}$ whose faces are:

$$\left\{W \subseteq V \frac{x_1^{a_1}\dots x_n^{a_n}}{\prod_{x_i \in W} x_i} \in I\right\}.$$

Theorem 2.6. [3, Theorem 2.2] *Given a monomial ideal I in $R = \mathbb{K}[x_1, \dots, x_n]$, the multigraded Betti numbers of I are*

$$\beta_{i,\alpha}(I) = dim_{\mathbb{K}} H_{i-1}(K^\alpha(I);\mathbb{K}) \text{ for } i \geq 0 \text{ and } \alpha \in \mathbb{N}^n.$$

For $d \in \mathbb{N}$the ith Betti number of degree d of I is

$$\beta_{i,d}(I) = \sum_{|\alpha|=d} \beta_{i,\alpha}(I),$$

where $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

Lemma 2.7. [37, Corollary 18.7] *Suppose that*

$$0 \longrightarrow U \longrightarrow U' \longrightarrow U'' \longrightarrow 0$$

is a short exact sequence of graded finitely generated R-modules with homomorphisms of degree 0. Then $\text{reg}(U') \leq \max\{\text{reg}(U),\text{reg}(U'')\}$.

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Lemma 2.8. [37, Proposition 18.6] *Suppose that*

$$0 \longrightarrow U \longrightarrow U' \longrightarrow U'' \longrightarrow 0$$

is a short exact sequence of graded finitely generated R-modules with homomorphisms of degree 0. *If* $\operatorname{reg}(U') > \operatorname{reg}(U'')$, *then* $\operatorname{reg}(U) = \operatorname{reg}(U')$.

3. Bounds for the Regularity of Generalized Edge Ideals

In this section, we give some upper bounds for the regularity of powers of generalized edge ideals. The following lemma gives us a nice description of the powers of the generalized edge ideals in the presence of a leaf vertex.

Lemma 3.1. *Let G be a simple graph and* $x_i \in V(G)$ *a leaf vertex with* $N_G(x_i) = \{x_j\}$. *Suppose that* $I = I_g(G)$ *is a generalized edge ideal of G,* $J = I_g(G \setminus x_i)$. *Then for all* $s \geq 2$ *we have the following.*

(i) $(I^s, x_i^{w_{i,j}} x_j^{w_{j,i}}) = (J^s, x_i^{w_{i,j}} x_j^{w_{j,i}})$;

(ii) *If* $w_{j,i} \leq w_{j,t}$ *for all* $x_t \in N_G(x_i)$, *then* $(I^s : x_i^{w_{i,j}} x_j^{w_{j,i}}) = I^{s-1}$.

Proof. **(*i*)** Clearly we have $(J^s : x_i^{w_{i,j}} x_j^{w_{j,i}}) \subseteq (I^s : x_i^{w_{i,j}} x_j^{w_{j,i}})$. For the reverse inclusion, let $f \in G(I^s) \setminus G(J^s)$ be a monomial. Then x_i divides f, and hence $x_i^{w_{i,j}} x_j^{w_{j,i}} | f$ as $N_G(x_i) = \{x_j\}$. This shows that $f \in (J^s, x_i^{w_{i,j}} x_j^{w_{j,i}})$.

(*ii*) It is clear that $I^{s-1} \subseteq (I^s : x_i^{w_{i,j}} x_j^{w_{j,i}})$. Suppose $f \in G(I^s : x_i^{w_{i,j}} x_j^{w_{j,i}})$; then $f x_i^{w_{i,j}} x_j^{w_{j,i}} \in I^s$. We can write $f x_i^{w_{i,j}} x_j^{w_{j,i}} = e_{p_1} e_{p_2} \dots e_{p_s} h$ for some $e_{p_k} \in G(I_g(G))$, $1 \leq k \leq s$, and $h \in R$. If there exists some $k \in [s]$ such that $x_i \mid e_{p_k}$, then $e_{p_k} = x_i^{w_{i,j}} x_j^{w_{j,i}}$, and hence $f = e_{p_1} \dots e_{p_{k-1}} e_{p_{k+1}} \dots e_{p_s} h \in I^{s-1}$. Otherwise if $x_i \nmid e_{p_k}$ for all $k \in [s]$, where $[s] = \{1, \cdots, s\}$. Then $x_i^{w_{i,j}} | h$. Now if $x_j^{w_{j,i}} | h$, then we are through. Otherwise if $x_j | e_{p_k}$ for some $1 \leq k \leq s$, then $x_j^{w_{j,i}} | e_{p_k}$ by our assumption that, $w_{j,i} \leq w_{j,t}$ for all $x_t \in N_G(x_j)$. So again we have $f \in I^{s-1}$.

The next result gives a bound for the regularity of powers of generalized edge ideals in terms of the regularity of two different ideals obtained from $I_g(G)^s$.

Theorem 3.2. *Let G be a simple graph and* $x_i \in V(G)$ *a leaf vertex with* $N_G(x_i) = \{x_j\}$. *Suppose that* $I = I_g(G)$ *is a generalized edge ideal of G,* $J = I_g(G \setminus x_i)$, *and* $w_{j,i} \leq w_{j,t}$ *for all* $x_t \in N_G(x_j)$.

Then for all $s \geq 2$

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$$reg(I^s) \leq w_{i,j} + w_{j,i} + \max\{\text{reg}(I^{s-1}), reg(J^s) - 1\}$$

Proof. Consider the following exact sequence:

$$0 \to \frac{R}{\left(I^s : x_i^{w_{i,j}} x_j^{w_{j,i}}\right)}(-w_{i,j} - w_{j,i}) \to \frac{R}{I^s} \to \frac{R}{(I^s, x_i^{w_{i,j}} x_j^{w_{j,i}})} \to 0$$

By applying Lemma 2.7 and Lemma 3.1, we have

$$reg(I^s) \leq \max\{reg\left(I^s : x_i^{w_{i,j}} x_j^{w_{j,i}}\right) + w_{j,i} + w_{i,j}, reg\left(I^s, x_i^{w_{i,j}} x_j^{w_{j,i}}\right)\}$$

$$= \max\{reg(I^s) + w_{j,i} + w_{i,j}, reg\left(J^s, x_i^{w_{i,j}} x_j^{w_{j,i}}\right)\}.$$

Now by [23, Corollary 3.2] $reg(J^s, x_i^{w_{i,j}} x_j^{w_{j,i}}) \leq reg(J^s) + w_{j,i} + w_{i,j} - 1$. Hence,

$$reg(I^s) \leq \max\{reg(I^{s-1}) + w_{j,i} + w_{i,j}, reg(J^s) + w_{j,i} + w_{i,j} - 1\}$$

$$= w_{j,i} + w_{i,j} + \max\{reg(I^{s-1}), reg(J^s) - 1\}.$$

The following corollary follows directly from the previous result, which will be used to prove the main theorem in this section.

Corollary 3.3. *Let G be a simple graph such that the hypothesis given in Theorem 3.2 is satisfied. Then using the same notations, we have for all* $s \geq 2$

$$reg(I^s) \leq w_{i,j} + w_{j,i} + \max\{reg(I^{s-1}), reg(J^s)\}.$$

Now we present one of the main results of this article, which gives a bound for the regularity of powers of generalized edge ideal $I_g(G)$ in terms of the regularity of the ideal $I_g(G)$ and the regularity of various powers of the generalized edge ideals of a subgraph obtained from G.

Theorem 3.4. *Let G be a simple graph and* $x_i \in V(G)$ *a leaf vertex with* $N_G(x_i) = \{x_j\}$. *Suppose that* $I = I_g(G)$ *is a generalized edge ideal of G,* $J = I_g(G \backslash x_i)$, *and* $w_{j,i} \leq w_{j,t}$ *for all* $x_t \in N_G(x_j)$. *Then for all* $s \geq 2$

$$reg(I^s) \leq (s-1)(w_{i,j} + w_{j,i}) + \max\{reg(I), reg(J^2), reg(J^3), \ldots, reg(J^s)\}.$$

Proof.We shall proceed by induction on s. For $s = 2$, the result follows from Corollary 3.3. Let $s \geq 3$ and assume that the result is true for $s - 1$. Now, by Corollary 3.3 and from the induction hypothesis, we get

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$$reg(I^s) \leq w_{i,j} + w_{j,i} + \max\{reg(I^{s-1}), reg(J^s)\}$$

$$\leq w_{i,j} + w_{j,i} + \max\{(s-2)(w_{i,j} + w_{j,i}) + \max\{reg(I), reg(J^2), reg(J^3), \dots, reg(J^{s-1})\}, reg(J^s)\}$$

$$\leq w_{i,j} + w_{j,i} + \max\{(s-2)(w_{i,j} + w_{j,i}) + \max\{reg(I), reg(J^2), reg(J^3), \dots, reg(J^{s-1})\}, (s-2)(w_{i,j} + w_{j,i}) + reg(J^s)\}$$

$$= (s-1)(w_{i,j} + w_{j,i}) + \max\{reg(I), reg(J^2), reg(J^3), \dots, reg(J^s)\}.$$

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